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# The evaluation of matrix elements for non-canonical Weyl tableau basis states adapted to $U(n_1+n_2) \supset U(n_1) \times U(n_2)$

## II. Explicit formulae for the matrix elements

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**Summary.** In this second paper, we offer a new insight and much simpler expressions for matrix elements in terms of non-canonical Weyl tableau basis functions adapted to subgroup chain $U(n_1+n_2) \supset U(n_1) \times U(n_2)$. The matrix elements can be expressed through the product of $U(n_1+1)$ and $U(n_2+1)$ matrix elements times a factor $A$, so it is a "global" rather than a "segment value".

**Key words:** Weyl tableau – Matrix elements

## 1. Introduction

The unitary-group approach (UGA) represents not only the simplest but also the most efficient procedure for CI (configuration interaction) problems in many-electron systems. It is based on earlier developments in the nuclear many-body problem, and on representation theory of compact Lie groups due to Moshinsky [1], Gelfand and Tsetlin [2], Nagel and Moshinsky [3], Baird and Biedenharn [4], Louck and Galbraith [5] and others.

The UGA to CI problems exploits the fact that the spin-independent many-electron Hamiltonian may be expressed as a bilinear form of the $U(n)$ generators. Hence, the CI matrix elements can be evaluated as linear combinations of the appropriate orbital integrals, where the coefficients are given in terms of the matrix elements of the $U(n)$ generators and their products between the canonical Gelfand bases.

However, in spite of the simplicity of the unitary group approach in principle, the explicit expressions of the generator matrix elements in Gelfand bases are rather complex and, if applied directly, might result in a rather inefficient computational scheme. Therefore, for the purpose of simple many-electron problems, Paldus [6] and Shavitt [7] have presented an effective and elegant simplified formalism based on the Paldus array. They showed that, although for an arbitrary column irreducible representation of $U(n)$ the questions of basis generation and generator matrix element evaluation are rather complex, there is a simple and compact solution of both of these problems for two-column irreducible representations, and the graphical unitary group approach (GUGA) results.

Recently, we have presented a Weyl type graphical method [8] for evaluating the matrix elements of $U(n)$ generators as well as products of generators, which is an extension of Harter's jawbone counting formula for elementary generators $E_{i,i-1}$ [9]. Later, by considering the transformation properties of the generators of $U(2n)$ and applying the Wigner–Eckart theorem repeatedly, we presented simple closed expressions for the generator matrix elements of $U(2n)$ in a non-canonical Weyl tableau bases, symmetry adapted to the group chain $U(2n) \supset U(2) \times U(n)$ [10]. These expressions are required for CI dealing with spin-orbit coupling. Another non-canonical group chain that is also important in the CI problem, is

$$U(n) \supset U(n_1) \times U(n_2), \quad n = n_1 + n_2. \tag{1}$$

Recently, a complete derivation of the $U(n)$ generator matrix elements in the non-canonical bases adapted to the group chain (1) was presented by Gould and Paldus [11] from the viewpoint of the Green–Gould characteristic identities for $GL(n)$ [12]. They obtained the expressions for the fundamental $U(n) : U(n_1) \times U(n_2)$ reduced Wigner coefficients and matrix elements. Paldus et al. [13] dealt with the same many-electron system partitioning in a Clifford algebra unitary group approach by several different methods, namely the permutation-orthogonalization method, the $U(n)$ Clebsch–Gordan coefficient method, and the linear algebraic equation method. A special case of partitioning is also employed in the particle-hole formalism, which was examined by Paldus and Boyle [14].

It is our aim in this paper to derive the detailed formulae of the generator matrix elements in the Weyl tableau bases adapted to the group chain (1). This derivation offers a new and useful viewpoint and gives flexibility to the formalism. The non-trivial case is given by the product of $U(n_1 + 1)$ and $U(n_2 + 1)$ matrix elements times a factor $A$. So it is "global" rather than a "segment values" solution. It is worthwhile to point out that the factor $A$ is somewhat similar to the "coupling segment value" of [11] and the "link segment" of the p–h formalism in [14], and essentially represents a $SU(2)$ $6-j$ symbol. In the previous paper of this series [15], explicit formulae for the subduction coefficients have been derived from which the final results will now be obtained.

## 2. Fundamentals

It is well known that the $U(n)$ generator matrix elements in the non-canonical Weyl tableau bases can be evaluated from those in the canonical Weyl tableau bases. The transformation reads:

$$\left\langle \begin{matrix}[V];[V_1'][V_2'] \\ W_1', W_2'\end{matrix} \middle| E_{ij} \middle| \begin{matrix}[V];[V_1][V_2] \\ W_1, W_2\end{matrix} \right\rangle$$
$$= \sum_{W,W'} \left\langle \begin{matrix}[V] \\ W'\end{matrix} \middle| \begin{matrix}[V_1'][V_2'] \\ W_1' W_2'\end{matrix} \right\rangle \left\langle \begin{matrix}[V] \\ W\end{matrix} \middle| \begin{matrix}[V_1][V_2] \\ W_1 W_2\end{matrix} \right\rangle \left\langle \begin{matrix}[V] \\ W'\end{matrix} \middle| E_{ij} \middle| \begin{matrix}[V] \\ W\end{matrix} \right\rangle. \tag{2}$$

The term on the left-hand side is the matrix element for the non-canonical Weyl tableau bases, the last term on the right-hand side is the matrix element in canonical Weyl tableau bases, and the first two terms are the subduction coefficients between the non-canonical and canonical Weyl tableau bases. Throughout we adopt the same notation as in [15].

Because of the Hermiticity relation,

$$E_{ij}^{+}=E_{ji}, \tag{3}$$

so we may confine our discussion to the lowering generators $E_{ij}$ with $i>j$. By considering the transformation properties of the $U(n)$ generators with respect to the subgroup imbedding of group chain (1), we obtain the following relations.

(a) In the first case of $1 \leqslant i, j \leqslant n_1$, the generators $E_{ij}$ are just the generators of subgroup $U(n_1)$ and we have

$$\left\langle \begin{matrix}[V];[V_1'][V_2'] \\ W_1', W_2'\end{matrix} \middle| E_{ij} \middle| \begin{matrix}[V];[V_1][V_2] \\ W_1, W_2\end{matrix} \right\rangle = \left\langle \begin{matrix}[V_1'] \\ W_1'\end{matrix} \middle| E_{ij} \middle| \begin{matrix}[V_1] \\ W_1\end{matrix} \right\rangle \left\langle \begin{matrix}[V_2'] \\ W_2'\end{matrix} \middle| \begin{matrix}[V_2] \\ W_2\end{matrix} \right\rangle$$
$$= \left\langle \begin{matrix}[V_1'] \\ W_1'\end{matrix} \middle| E_{ij} \middle| \begin{matrix}[V_1] \\ W_1\end{matrix} \right\rangle \delta_{V_1',V_1}\delta_{V_2',V_2}\delta_{W_2',W_2}. \tag{4}$$

(b) For the other special case, $n_1+1 \leqslant i, j \leqslant n=n_1+n_2$, we have

$$\left\langle \begin{matrix}[V];[V_1'][V_2'] \\ W_1', W_2'\end{matrix} \middle| E_{ij} \middle| \begin{matrix}[V];[V_1][V_2] \\ W_1, W_2\end{matrix} \right\rangle = \left\langle \begin{matrix}[V_1'] \\ W_1'\end{matrix} \middle| \begin{matrix}[V_1] \\ W_1\end{matrix} \right\rangle \left\langle \begin{matrix}[V_2'] \\ W_2'\end{matrix} \middle| E_{ij} \middle| \begin{matrix}[V_2] \\ W_2\end{matrix} \right\rangle$$
$$= \left\langle \begin{matrix}[V_2'] \\ W_2'\end{matrix} \middle| E_{ij} \middle| \begin{matrix}[V_2] \\ W_2\end{matrix} \right\rangle \delta_{V_1',V_1}\delta_{V_2',V_2}\delta_{W_1',W_1}. \tag{5}$$

(c) The non-trivial cases are those where $i$, $j$ refer to orbitals of different subgroups. In the case of the lowering generators, $i$ belongs to the subgroup $U(n_2)$ and $j$ to $U(n_1)$, namely,

$$n_1+1 \leqslant i \leqslant n_1+n_2=n$$
$$1 \leqslant j \leqslant n_1.$$

In such cases, the generators $E_{ij}$ transform as the tensor operators of contragradient vector operators of $U(n_1)$ and vector operators of $U(n_2)$. If $[V_1]\times[V_2]$ denotes the irreducible representations of $U(n_1)\times U(n_2)$, and the explicit labels $[\lambda_1^{(p)}, \lambda_2^{(p)}]$ are introduced instead of $[V_p]$, where $\lambda_q^{(p)}$ is the number of boxes in the $q$th column of the Young diagram $[V_p]$, then all the four shifting effects are as follows:

$$\begin{aligned}
&1) \qquad [\lambda_1^{(1)}-1, \lambda_2^{(1)}]\times[\lambda_1^{(2)}+1, \lambda_2^{(2)}]\\
&2) \qquad [\lambda_1^{(1)}-1, \lambda_2^{(1)}]\times[\lambda_1^{(2)}, \lambda_2^{(2)}+1]\\
&3) \qquad [\lambda_1^{(1)}, \lambda_2^{(1)}-1]\times[\lambda_1^{(2)}+1, \lambda_2^{(2)}]\\
&4) \qquad [\lambda_1^{(1)}, \lambda_2^{(1)}-1]\times[\lambda_1^{(2)}, \lambda_2^{(2)}+1].
\end{aligned} \tag{6}$$

Applying the $U(n_1) \times U(n_2)$ Wigner–Eckart theorem to the non-canonical matrix elements, we may write

$$\left\langle \begin{matrix}[V];[V_1'][V_2'] \\ W_1', W_2'\end{matrix} \middle| E_{ij} \middle| \begin{matrix}[V];[V_1][V_2] \\ W_1, W_2\end{matrix} \right\rangle$$

$$= \langle [V];[V_1'],[V_2'] \| E \| [V];[V_1],[V_2] \rangle \left\langle \begin{matrix}[V_1'] \\ W_1'\end{matrix} \middle| \bar{j}; \begin{matrix}[V_1] \\ W_1\end{matrix} \right\rangle \left\langle \begin{matrix}[V_2'] \\ W_2'\end{matrix} \middle| i; \begin{matrix}[V_2] \\ W_2\end{matrix} \right\rangle, \quad (7)$$

where the first term on the right-hand side is the $U(n_1) \times U(n_2)$ reduced matrix element, being dependent only on the Young diagrams of $[V]$, $[V_1']$, $[V_2']$, $[V_1]$ and $[V_2]$, the second term is a $U(n_1)$ contragradient vector coupling coefficient, and the third term is a $U(n_2)$ vector coupling coefficient with $\bar{j}$ and $i$ being the corresponding unit bases in terms of Weyl tableau bases.

Now, in order to eliminate the reduced matrix elements in Eq. (7), we consider the non-canonical matrix element of a pertinent generator $E_{kl}$ where $l$ is in the range $(1, n_1)$ and $k$ in the range $(n_1+1, n_1+n_2)$. If the indices $k$, $l$ are chosen appropriately (see Sect. 3) the following relation can be exploited

$$\left\langle \begin{matrix}[V];[V_1'] \quad [V_2'] \\ W_{1m}', W_{2m}'\end{matrix} \middle| E_{kl} \middle| \begin{matrix}[V];[V_1] \quad [V_2] \\ W_{1m}, W_{2m}\end{matrix} \right\rangle$$

$$= \langle [V];[V_1'],[V_2'] \| E \| [V];[V_1],[V_2] \rangle \left\langle \begin{matrix}[V_1'] \\ W_{1m}'\end{matrix} \middle| \bar{l}; \begin{matrix}[V_1] \\ W_{1m}\end{matrix} \right\rangle \left\langle \begin{matrix}[V_2'] \\ W_{2m}'\end{matrix} \middle| k; \begin{matrix}[V_2] \\ W_{2m}\end{matrix} \right\rangle, \quad (8)$$

where $|W_{1m}'\rangle$, $|W_{2m}'\rangle$, $|W_{1m}\rangle$ and $|W_{2m}\rangle$ are the corresponding Weyl tableau bases, which yield non-trivial results. Combining Eqs. (7) and (8), we obtain the following equation:

$$\left\langle \begin{matrix}[V];[V_1'][V_2'] \\ W_1', W_2'\end{matrix} \middle| E_{ij} \middle| \begin{matrix}[V];[V_1][V_2] \\ W_1, W_2\end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix}[V];[V_1'] \quad [V_2'] \\ W_{1m}', W_{2m}'\end{matrix} \middle| E_{kl} \middle| \begin{matrix}[V];[V_1] \quad [V_2] \\ W_{1m}, W_{2m}\end{matrix} \right\rangle \left\langle \begin{matrix}[V_1'] \\ W_1'\end{matrix} \middle| \bar{j}; \begin{matrix}[V_1] \\ W_1\end{matrix} \right\rangle \left\langle \begin{matrix}[V_2'] \\ W_2'\end{matrix} \middle| i; \begin{matrix}[V_2] \\ W_2\end{matrix} \right\rangle$$

$$\times \left[ \left\langle \begin{matrix}[V_1'] \\ W_{1m}'\end{matrix} \middle| \bar{l}; \begin{matrix}[V_1] \\ W_{1m}\end{matrix} \right\rangle \left\langle \begin{matrix}[V_2'] \\ W_{2m}'\end{matrix} \middle| k; \begin{matrix}[V_2] \\ W_{2m}\end{matrix} \right\rangle \right]^{-1}. \quad (9)$$

Using the property of $E_{n_1+1,r}$ $(r = 1, 2, \ldots, n_1)$ forming a contragradient vector operator of $U(n_1)$, and $E_{t,n_1+n_2+1}$ $(t = n_1+1, n_1+2, \ldots, n_1+n_2)$ forming a vector operator of $U(n_2)$, we apply the Wigner–Eckart theorem again and obtain

$$\left\langle \begin{matrix}[V];[V_1'][V_2'] \\ W_1', W_2'\end{matrix} \middle| E_{ij} \middle| \begin{matrix}[V];[V_1] \; [V_2] \\ W_1, W_2\end{matrix} \right\rangle$$

$$= A \left\langle \begin{matrix}[V_{n_1+1}] \\ [V_1'] \\ W_1'\end{matrix} \middle| E_{n_1+1,j} \middle| \begin{matrix}[V_{n_1+1}] \\ [V_1] \\ W_1\end{matrix} \right\rangle \left\langle \begin{matrix}[V_{n_2+1}] \\ [V_2'] \\ W_2'\end{matrix} \middle| E_{i,n_1+n_2+1} \middle| \begin{matrix}[V_{n_2+1}] \\ [V_2] \\ W_2\end{matrix} \right\rangle, \quad (10)$$

where

$$A = \left\langle \begin{matrix} [V]; [V_1'] & [V_2'] \\ W_{1m}', & W_{2m}' \end{matrix} \middle| E_{kl} \middle| \begin{matrix} [V]; [V_1] & [V_2] \\ W_{1m}, & W_{2m} \end{matrix} \right\rangle$$

$$\times \left[ \left\langle \begin{matrix} [V_{n_1+1}] \\ [V_1'] \\ W_{1m}' \end{matrix} \middle| E_{n_1+1,l} \middle| \begin{matrix} [V_{n_1+1}] \\ [V_1] \\ W_{1m} \end{matrix} \right\rangle \left\langle \begin{matrix} [V_{n_2+1}] \\ [V_2'] \\ W_{2m}' \end{matrix} \middle| E_{k,n_1+n_2+1} \middle| \begin{matrix} [V_{n_2+1}] \\ [V_2] \\ W_{2m} \end{matrix} \right\rangle \right]^{-1}$$

$$= M_T \cdot [M_1 \cdot M_2]^{-1}. \tag{11}$$

In Eqs. (10–11), the selection of the irreducible representations $[V_{n_1+1}]$ of $U(n_1+1)$ and $[V_{n_2+1}]$ of $U(n_2+1)$ should satisfy the following decomposition conditions:

$$\begin{aligned} &[V_{n_1+1}] \supset [V_1] \\ &[V_{n_1+1}] \supset [V_1'] \\ &[V_{n_2+1}] \supset [V_2] \\ &[V_{n_2+1}] \supset [V_2']. \end{aligned} \tag{12}$$

It is obvious that the $[V_{n_1+1}]$ and $[V_{n_2+1}]$ satisfying Eq. (12) are not unique. We now make a special choice for the four cases of different shifts using the criterion of maximum symmetry in the final formulae

$$\begin{aligned} &[V_{n_1+1}] = [V_1'] + [1, 1] \\ &[V_{n_2+1}] = [V_2']. \end{aligned} \tag{13}$$

## 3. Detailed derivation

We now direct our attention to the remaining problem of how to obtain the value of $A$ in Eqs. (10–11). It should be noted that, for the definite choice of Eq. (13), $A$ depends only on $[V]$, $[V_1]$ and $[V_2]$ for a given shift case. We consider the special shift of $[\lambda_1^{(1)}, \lambda_2^{(1)} - 1] \times [\lambda_1^{(2)} + 1, \lambda_2^{(2)}]$ as an example. According to Eq. (11), the calculation of $A$ is that of three generator matrix elements, i.e.

$$A = M_T/(M_1 M_2). \tag{14}$$

At first, we consider the simplest case, namely,

$$\begin{aligned} &[V_1] = [\lambda_1^{(1)}, 1] \\ &[V_2] = [\lambda_1^{(2)}, 0] \\ &[V_1'] = [\lambda_1^{(1)}, 0] \\ &[V_2'] = [\lambda_1^{(2)} + 1, 0]. \end{aligned} \tag{15}$$

By virtue of the Littlewood–Richardson rule, we know that

$$[V] = [\lambda_1, \lambda_2],$$

where $\lambda_1$, $\lambda_2$ satisfy the following relations

$$\begin{aligned} &\lambda_1 = \lambda_1^{(1)} + \lambda_1^{(2)} - \Delta s \\ &\lambda_2 = 1 + \Delta s \end{aligned} \tag{16}$$

and

$$\Delta s = 0, 1, 2, \ldots, d_2 - 1 \quad \text{for } d_1 \geqslant d_2$$
$$\Delta s = 0, 1, 2, \ldots, d_1 - 1 \quad \text{for } d_1 < d_2. \tag{17}$$

In Eq. (17), $d_i$ refers to the axial distance between the last box of each column in $[V_i]$, that is

$$d_i = \lambda_1^{(i)} - \lambda_2^{(i)} + 1. \tag{18}$$

Now, take

$$E_{kl} = E_{n_1+1,n_1} \tag{19}$$

$$\left|\begin{matrix}[V_1'] \\ W_{1m}'\end{matrix}\right\rangle = \left|\begin{matrix}1 \\ 2 \\ \vdots \\ \lambda_1^{(1)}\end{matrix}\right\rangle \qquad \left|\begin{matrix}[V_1] \\ W_{1m}\end{matrix}\right\rangle = \left|\begin{matrix}1 & n_1 \\ 2 & \\ \vdots & \\ \lambda_1^{(1)} & \end{matrix}\right\rangle \tag{20}$$

$$\left|\begin{matrix}[V_2'] \\ W_{2m}'\end{matrix}\right\rangle = \left|\begin{matrix}n_1+1 \\ n_1+2 \\ n_1+3 \\ \vdots \\ n_1+\lambda_1^{(2)}+1\end{matrix}\right\rangle \qquad \left|\begin{matrix}[V_2] \\ W_{2m}\end{matrix}\right\rangle = \left|\begin{matrix}n_1+2 \\ n_1+3 \\ \vdots \\ n_1+\lambda_1^{(2)}+1\end{matrix}\right\rangle,$$

$$[V_{n_1+1}] = [\lambda_1^{(1)}+1, 1]$$
$$[V_{n_2+1}] = [\lambda_1^{(2)}+1, 0]. \tag{21}$$

Substituting Eqs. (19–21) into Eq. (11) and using the method of [8], we obtain

$$M_1 = \left\langle \begin{matrix}[V_{n_1+1}] \\ [V_1'] \\ W_{1m}'\end{matrix} \middle| E_{n_1+1,n_1} \middle| \begin{matrix}[V_{n_1+1}] \\ [V_1] \\ W_{1m}\end{matrix} \right\rangle = \sqrt{\frac{d_1}{d_1+1}}, \tag{22}$$

$$M_2 = \left\langle \begin{matrix}[V_{n_2+1}] \\ [V_2'] \\ W_{2m}'\end{matrix} \middle| E_{n_1+1,n_1+n_2+1} \middle| \begin{matrix}[V_{n_2+1}] \\ [V_2] \\ W_{2m}\end{matrix} \right\rangle = (-1)^{d_2-1}. \tag{23}$$

The most complicated step is the calculation of $M_T$, which depends on the different $\Delta s$. Combining Eqs. (2), (11), (14) and (19–20), we finally obtain

$$M_T = \left\langle \begin{matrix}[V]; [V_1'] & [V_2'] \\ & W_{1m}', W_{2m}'\end{matrix} \middle| E_{n_1+1,n_1} \middle| \begin{matrix}[V]; [V_1] & [V_2] \\ & W_{1m}, W_{2m}\end{matrix} \right\rangle_{\Delta s}$$
$$= \sum_{W_m, W_m'} M' \left\langle \begin{matrix}[V] \\ W_m'\end{matrix} \middle| \begin{matrix}[V_1'] [V_2'] \\ W_{1m}' W_{2m}'\end{matrix} \right\rangle_{\Delta s} \left\langle \begin{matrix}[V] \\ W_m\end{matrix} \middle| \begin{matrix}[V_1] [V_2] \\ W_{1m} W_{2m}\end{matrix} \right\rangle_{\Delta s}, \tag{24}$$

where

$$M' = \left\langle \begin{matrix}[V] \\ W_m'\end{matrix} \middle| E_{n_1+1,n_1} \middle| \begin{matrix}[V] \\ W_m\end{matrix} \right\rangle_{\Delta s}. \tag{25}$$

In order to go a step further, three things should be pointed out:

a) According to the selection rule for non-zero elementary generator matrix element in $U(n)$ (see [8]), the Weyl tableau bases $|W_m\rangle$ and $|W'_m\rangle$ in the preceding equations must be the same except in one box, which is covered by $n_1$ in $|W_m\rangle$ and by $n_1+1$ in $|W'_m\rangle$. Thus, there is a one to one correspondence relation between $|W_m\rangle$ and $|W'_m\rangle$, and furthermore, the desired $|W'_m\rangle$ can be easily determined from $|W_m\rangle$.
b) In our special case of Eqs. (19–20), the different box is the first box of the second column.
c) $M'$ takes the value 1 which is independent of different $\Delta s$.

$$M' = \left\langle \begin{matrix}[V]\\ W'_m\end{matrix} \middle| E_{n_1+1,n_1} \middle| \begin{matrix}[V]\\ W_m\end{matrix} \right\rangle = 1. \tag{26}$$

(In the special case where $\lambda_1^{(1)} = n_1$, $M_1$ will take the value 1, but $M'$ will take the value $[(d_1+1)/d_1]^{1/2}$. Now, $d_1 = n_1$, hence the formula of $A$ remains unchanged.)
Thus, Eq. (24) simplifies to

$$M_T = \sum_{W_m} \left\langle \begin{matrix}[V]\\ W'_m\end{matrix} \middle| \begin{matrix}[V'_1]\,[V'_2]\\ W'_{1m}\,W'_{2m}\end{matrix} \right\rangle_{\Delta s} \left\langle \begin{matrix}[V]\\ W_m\end{matrix} \middle| \begin{matrix}[V_1]\,[V_2]\\ W_{1m}\,W_{2m}\end{matrix} \right\rangle_{\Delta s}. \tag{27}$$

Now, we evaluate Eq. (27) for the first several $\Delta s$ by using the results of [15]. For $\Delta s = 0$, the summation includes just one, i.e. $\binom{d_2-1}{0}$, term and we obtain

$$M_T = \{(d_1+d_2)/[d_2(d_1+1)]\}^{1/2}.$$

For $\Delta s = 1$, the summation will include $\binom{d_2-1}{1}$ terms and we obtain

$$M_T = \{2(d_1+d_2-1)/[d_2(d_1+1)]\}^{1/2}.$$

For $\Delta s = 2$, $\binom{d_2-1}{2}$ terms are included and we obtain

$$M_T = \{3(d_1+d_2-2)/[d_2(d_1+1)]\}^{1/2}.$$

Then, induction yields the final result:

$$M_T = \{(1+\Delta s)(d_1+d_2-\Delta s)/[d_2(d_1+1)]\}^{1/2}, \tag{28}$$

where the following identity has been used:

$$\begin{aligned} &\sum_{j_1=2}^{d_2-(\Delta s-1)} \frac{1}{(d_1+j_1-2)(d_1+j_1-3)} \left\{ \sum_{j_2=j_1+1}^{d_2-(\Delta s-2)} \frac{1}{(d_1+j_2-4)(d_1+j_2-5)} \right.\\ &\quad \left. \times\left\{ \cdots \left\{ \sum_{j_{\Delta s}=j_{\Delta s-1}+1}^{d_2} \frac{1}{(d_1+j_{\Delta s}-2\Delta s)(d_1+j_{\Delta s}-2\Delta s-1)} \right\} \cdots \right\} \right\} + \delta(\Delta s, 0)\\ &= \frac{(d_1-\Delta s-1)!(d_1+d_2-2\Delta s-1)!(d_2-1)!}{(d_2-\Delta s-1)!(d_1-1)!\,\Delta s!(d_1+d_2-\Delta s-1)!}. \end{aligned} \tag{28a}$$

It should be noted that all the parameters $d_i$ and $\Delta s$ must satisfy the selection rules of Eq. (17), otherwise, a negative integer factorial will appear.

Substituting Eqs. (22–23) and (28) into Eq. (14), we obtain

$$A=(-1)^{d_2-1}\sqrt{\frac{(1+\Delta s)(d_1+d_2-\Delta s)}{d_1 d_2}}\,. \tag{29}$$

In the following, we extend the result of Eq. (29) to more general cases. First we consider

$$\begin{aligned}[][V_1]&=[\lambda_1^{(1)},\lambda_2^{(1)}]\\ [V_1']&=[\lambda_1^{(1)},\lambda_2^{(1)}-1].\end{aligned} \tag{30}$$

In this situation, as long as we take

$$\left|\begin{matrix}[V_1']\\ W_{1m}'\end{matrix}\right\rangle=\left|\begin{matrix}1 & 1\\ 2 & 2\\ \vdots & \vdots\\ \lambda_2-1 & \lambda_2-1\\ \lambda_2 & \\ \vdots & \\ \lambda_1 & \end{matrix}\right\rangle \qquad \left|\begin{matrix}[V_1']\\ W_{1m}\end{matrix}\right\rangle=\left|\begin{matrix}1 & 1\\ 2 & 2\\ \vdots & \vdots\\ \lambda_2-1 & \lambda_2-1\\ \lambda_2 & n_1\\ \vdots & \\ \lambda_1 & \end{matrix}\right\rangle \tag{31}$$

where all the $\lambda_i$ refer to $\lambda_i^{(1)}$ and $[V_{n_1+1}]=[\lambda_1^{(1)}+1,\lambda_2^{(1)}]$, then all the results obtained above retain the same form.

Next, we consider the most general case

$$\begin{aligned}[][V_2]&=[\lambda_1^{(2)},\lambda_2^{(2)}]\\ [V_2']&=[\lambda_1^{(2)}+1,\lambda_2^{(2)}].\end{aligned} \tag{32}$$

Now, we take

$$\left|\begin{matrix}[V_2']\\ W_{2m}'\end{matrix}\right\rangle=\left|\begin{matrix}n_1+1 & n_1+1\\ \vdots & \vdots\\ n_1+\lambda_2 & n_1+\lambda_2\\ n_1+\lambda_2+1 & \\ n_1+\lambda_2+2 & \\ n_1+\lambda_2+3 & \\ \vdots & \\ n_1+\lambda_1+1 & \end{matrix}\right\rangle \qquad \left|\begin{matrix}[V_2]\\ W_{2m}\end{matrix}\right\rangle=\left|\begin{matrix}n_1+1 & n_1+1\\ \vdots & \vdots\\ n_1+\lambda_2 & n_1+\lambda_2\\ n_1+\lambda_2+2 & \\ n_1+\lambda_2+3 & \\ \vdots & \\ n_1+\lambda_1+1 & \end{matrix}\right\rangle \tag{33}$$

and

$$\begin{aligned}[][V_{n_2+1}]&=[\lambda_1+1,\lambda_2]\\ E_{kl}&=E_{n_1+\lambda_2+1,n_1}.\end{aligned} \tag{34}$$

In Eqs. (33–34), all the $\lambda_i$ refer to $\lambda_i^{(2)}$. In this situation, the only thing which has changed is the value of $M'$ in Eq. (26). Here,

$$M'=\left\langle\begin{matrix}[V]\\ W_m'\end{matrix}\right|E_{n_1+\lambda_2^{(2)}+1,n_1}\left|\begin{matrix}[V]\\ W_m\end{matrix}\right\rangle=(-1)^{\lambda_2^{(2)}}. \tag{35}$$

So, the final result for the general case of $[V_1]$ and $[V_2]$ is

$$\begin{aligned} A &= (-1)^{\lambda_2^{(2)}+d_2-1}\sqrt{\frac{(1+\Delta s)(d_1+d_2-\Delta s)}{d_1 d_2}} \\ &= (-1)^{\lambda_1^{(2)}}\sqrt{\frac{(1+\Delta s)(d_1+d_2-\Delta s)}{d_1 d_2}}\,. \end{aligned} \tag{36}$$

As to the other three cases of shifts, the results can be obtained by a similar procedure, the final expressions are given in the next section.

## 4. The final expressions

$$\left\langle \begin{matrix} [V];[V_1'][V_2'] \\ W_1', W_2' \end{matrix} \middle| E_{ij} \middle| \begin{matrix} [V];[V_1][V_2] \\ W_1, W_2 \end{matrix} \right\rangle$$

$$= A \left\langle \begin{matrix} [V_{n_1+1}] \\ [V_1'] \\ W_1' \end{matrix} \middle| E_{n_1+1,j} \middle| \begin{matrix} [V_{n_1+1}] \\ [V_1] \\ W_1 \end{matrix} \right\rangle \left\langle \begin{matrix} [V_{n_2+1}] \\ [V_2'] \\ W_2' \end{matrix} \middle| E_{i,n_1+n_2+1} \middle| \begin{matrix} [V_{n_2+1}] \\ [V_2] \\ W_2 \end{matrix} \right\rangle, \tag{37}$$

$$[V_{n_1+1}] = [V_1'] + [1,1]$$

$$[V_{n_2+1}] = [V_2'].$$

For the first shift case, $[\lambda_1^{(1)}-1, \lambda_2^{(1)}] \times [\lambda_1^{(2)}+1, \lambda_2^{(2)}]$:

$$A = (-1)^{\lambda_1^{(2)}+\Delta s}\sqrt{\frac{(d_1-\Delta s-1)(d_2-\Delta s)}{d_1 d_2}} \tag{38}$$

$$\Delta s = 0, 1, \ldots, d_1-2, \quad d_2 \geqslant d_1$$

$$\Delta s = 0, 1, \ldots, d_2-1, \quad d_2 < d_1.$$

For the second shift case, $[\lambda_1^{(1)}-1, \lambda_2^{(1)}] \times [\lambda_1^{(2)}, \lambda_2^{(2)}+1]$:

$$A = (-1)^{\lambda_2^{(2)}}\sqrt{\frac{(d_1+d_2-\Delta s-1)\,\Delta s}{d_1 d_2}} \tag{39}$$

$$\Delta s = 0, 1, \ldots, d_1-1, \quad d_2 \geqslant d_1$$

$$\Delta s = 0, 1, \ldots, d_2-1, \quad d_2 < d_1.$$

For the third shift case, $[\lambda_1^{(1)}, \lambda_2^{(1)}-1] \times [\lambda_1^{(2)}+1, \lambda_2]$:

$$A = (-1)^{\lambda_1^{(2)}}\sqrt{\frac{(d_1+d_2-\Delta s)(1+\Delta s)}{d_1 d_2}} \tag{40}$$

$$\Delta s = 0, 1, \ldots, d_1-1, \quad d_2 \geqslant d_1$$

$$\Delta s = 0, 1, \ldots, d_2-1, \quad d_2 < d_1.$$

For the last shift case, $[\lambda_1^{(1)}, \lambda_2^{(1)} - 1] \times [\lambda_1^{(2)}, \lambda_2^{(2)} + 1]$:

$$A = (-1)^{\lambda_2^{(2)} + \Delta s} \sqrt{\frac{(d_1 - \Delta s)(d_2 - \Delta s - 1)}{d_1 d_2}} \tag{41}$$

$$\Delta s = 0, 1, \ldots, d_1 - 1, \quad d_2 > d_1$$

$$\Delta s = 0, 1, \ldots, d_2 - 2, \quad d_2 \leqslant d_1 .$$

Finally, we give two examples to illustrate these formulae.

Example (1), for the third shift case, $U(6) \supset U(3) \times U(3)$,

$$\left\langle \begin{array}{ccccc} & 1 & & 4 & 6 \\ [2^3]; & 3 & \otimes & 5 & \\ & & & 6 & \end{array} \right| E_{62} \left| \begin{array}{cccccc} & 1 & 2 & & 4 & 6 \\ [2^3]; & 3 & & \otimes & 5 & \\ & & & & & \end{array} \right\rangle_{\Delta s = 1}$$

$$= (-1)^{\lambda_1^{(2)}} \sqrt{\frac{(d_1 + d_2 - \Delta s)(1 + \Delta s)}{d_1 d_2}} \left\langle \begin{array}{cc} 1 & 4 \\ 3 & \\ 4 & \end{array} \right| E_{42} \left| \begin{array}{cc} 1 & 2 \\ 3 & \\ 4 & \end{array} \right\rangle \left\langle \begin{array}{cc} 4 & 6 \\ 5 & \\ 6 & \end{array} \right| E_{67} \left| \begin{array}{cc} 4 & 6 \\ 5 & \\ 7 & \end{array} \right\rangle$$

$$= (-1)^2 \sqrt{\frac{(2 + 2 - 1)(1 + 1)}{2 \cdot 2}} \cdot \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{2}{3}} = \sqrt{\frac{1}{2}} .$$

Example (2) for the last shift case, $U(6) \supset U(3) \times U(3)$,

$$\left\langle \begin{array}{cccccc} & 1 & 3 & & 4 & 5 \\ [2^4, 1]; & 2 & & \otimes & 5 & 6 \\ & 3 & & & 6 & \end{array} \right| E_{52} \left| \begin{array}{cccccc} & 1 & 2 & & 4 & 6 \\ [2^4, 1]; & 2 & 3 & \otimes & 5 & \\ & 3 & & & 6 & \end{array} \right\rangle_{\Delta s = 1}$$

$$= (-1)^{\lambda_2^{(2)} + \Delta s} \sqrt{\frac{(d_1 - \Delta s)(d_2 - \Delta s - 1)}{d_1 d_2}}$$

$$\times \left\langle \begin{array}{cc} 1 & 3 \\ 2 & 4 \\ 3 & \\ 4 & \end{array} \right| E_{42} \left| \begin{array}{cc} 1 & 2 \\ 2 & 3 \\ 3 & \\ 4 & \end{array} \right\rangle \left\langle \begin{array}{cc} 4 & 5 \\ 5 & 6 \\ 6 & \end{array} \right| E_{57} \left| \begin{array}{cc} 4 & 6 \\ 5 & 7 \\ 6 & \end{array} \right\rangle$$

$$= (-1)^{1+1} \sqrt{\frac{(2 - 1)(3 - 1 - 1)}{2 \cdot 3}} \cdot (-1) \cdot \left( -\sqrt{\frac{3}{2}} \right) = \frac{1}{2} .$$

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